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Optimal Pole Placement in Time-Dependent Linear Systems

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In this paper a problem of optimal modal control for general linear, time-dependent systems is posed and solved. We review the recent modal decomposition of the general linear system. We then pose the problem of controlling one unstable mode, with a specified new Lyapunov exponent, but minimizing the mean-square gain function. The method is applied to the shallow-angle re-entry of a Delta Clipper-like vehicle, and the modal decomposition leads to both stability information and characterization of how the trajectory is unstable. The unstable Lyapunov exponent is moved so that the trajectory is stable, as verified by closed-loop Lyapunov exponent analysis. When implemented within the original nonlinear system, the controller can handle initial errors at the kilometer level. Only 4% modulation of the original drag factor is required for control.

I. Introduction

THE problem of control of constant coefficient linear systems is well understood, forming the basis of most of control theory. Control of time-dependent systems is not yet at the same level of understanding. Fairly comprehensive discussion of the time-periodic case have appeared: see Breakwell et al.[1] for the linear quadratic regulator for periodic systems and Calico and Wiesel[2] for pole placement in periodic systems. Recently, the current author has successfully produced the modal decomposition for a general time-dependent linear system[3] and introduced the first pole placement algorithm for such systems. In this paper we extend the method to optimal pole placement and will successfully control a shallow-angle re-entry trajectory by this new technique.

II. Modal Decomposition

A general dynamic system can be written as a vector set of differential equations

$$\dot{X} = f(X, t) \tag{1}$$

where X is termed the state vector. Introduce the small displacement $x(t) = X(t) - X_0(t)$ from a known trajectory $X_0(t)$. Then, to first order in small quantities, the displacement vector obeys the variational equations

$$\dot{x} = A(t)x = \left.\frac{\partial f}{\partial X}\right|_{X_0} x \tag{2}$$

As a set of linear equations, the variational equations are formally solved by the fundamental matrix $\Phi(t, t_0)$, which obeys

$$\dot{\Phi} = A(t)\Phi, \qquad \Phi(t_0, t_0) = I \tag{3}$$

For the remainder of this section we will briefly review the modal decomposition of Wiesel[3] without proof. The reader is encouraged to consult that article for somewhat fuller details. The stability of a general trajectory of a linear system is determined by the Lyapunov exponents. Over the finite time interval (t_f, t_0), these are the values

$$\lambda_i = \frac{1}{2(t_f - t_0)} \log \mu_i \tag{4}$$

where the μ_i are the singular values of $\Phi(t_f, t_0)$. This has been recognized by Goldhirsch et al.[4] Since we are not interested in either the infinitesimal limit, leading to local Lyapunov exponents, or the infinite time limit, leading to the classical Lyapunov exponents, we will refer to our λ_i as regional Lyapunov exponents over the finite time interval (t_0, t_f).

Over a finite arc of the trajectory, the regional Lyapunov exponents may be used to factor the dynamics into separate modes. The initial conditions $\mathcal{E}(t_0)$, the left singular value matrix of $\Phi(t_f, t_0)$, introduce a special set of basis solutions, obeying

$$\dot{\mathcal{E}}(t) = A(t)\mathcal{E}(t) - \mathcal{E}(t)J(t) \tag{5}$$

We note without proof that $e_i(t)$, the column vectors of $\mathcal{E}(t)$ are unit vectors at all times, and that $\mathcal{E}(t_f)$ is the right singular value matrix of $\Phi(t_f, t_0)$. The diagonal matrix $J(t)$ has entries

$$J_{ii}(t) = \sigma_i(t) = e_i(t) \cdot A(t)e_i(t) \tag{6}$$

Then, the coordinate transformation

$$x(t) = \mathcal{E}(t)y(t) \tag{7}$$

reduces Eq. (2) to

$$\dot{y}(t) = J(t)y(t) \tag{8}$$

a set of decoupled, time-dependent coefficient differential equations for the variables y and another set of linear equations (5) for the coordinate vectors $e(t)$. We will refer to y as the modal variables for the system and $\mathcal{E}(t)$ as the modal matrix. This is the general case of the modal transformation implicit in Floquet theory for time-periodic systems and used by Calico and Wiesel[2] for feedback control of periodic trajectories.

III. Optimal Modal Control

A linear dynamic system subject to a control system is usually written as

$$\dot{x} = A(t)x + B(t)u \tag{9}$$

where u is the vector of control variables and the matrix $B(t)$ apportions the control to the physical states that can be influenced. Introducing the modal variables, Eq. (9) becomes

$$\begin{aligned}\dot{y} &= J(t)y + \mathcal{E}^{-1}(t)B(t)u \\ &= \{J(t) + \mathcal{E}^{-1}(t)B(t)G(t)\}y\end{aligned} \tag{10}$$

the second line assuming that the control is the product of a gain matrix $G(t)$ with the modal state $u = G(t)y$.

As a very important special case, consider a situation where only one Lyapunov exponent λ_1 is greater than zero. The advantage of the modal separation is now clear: Since only y_1 states are growing with time, we need only feed back the one unstable state $y_1(t)$. The exponentially decaying behavior of $y_i(t)$, $i = 2, \ldots, N$ is deemed acceptable at the moment. Both the gain matrix and control vector then

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collapse to scalars, say, $G(t) = g(t)$, whereas $\mathcal{E}^{-1}(t)B(t) = \boldsymbol{c}(t)$ is a vector function of time. [Although this is a scalar control/scalar gain controller in the modal variables, when Eq. (7) is used to convert to the physical variables $\boldsymbol{x}$, it becomes a scalar control/vector gain controller. Controllers with a vector $\boldsymbol{u}$ can also be designed if one is willing to make definite statements about the relative importance of different control inputs in the scalar optimality condition.] To stabilize the trajectory against small displacements, the positive Lyapunov exponent must be changed into a negative one, without altering the stability characteristics of the other exponents. We wish to leave the other exponents unaltered, lest the control for the first mode inadvertently drive an uncontrolled mode unstable. This is the best that can be achieved without a theory that allows simultaneous placement of all the Lyapunov exponents.

Under these conditions, the system (10) becomes

$$\dot{y} = \begin{bmatrix} \sigma_1(t) + c_1(t)g(t) & 0 & \cdots & 0 \\ c_2(t)g(t) & \sigma_2(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ c_N(t)g(t) & 0 & \cdots & \sigma_N(t) \end{bmatrix} y \tag{11}$$

Consider first the unstable mode y_1, which now has an equation of motion

$$\dot{y}_1 = [\sigma_1(t) + c_1(t)g(t)]y_1 \tag{12}$$

Since $c_1(t)$ and $g(t)$ are certainly functions of time, this has solution

$$y_1(t) = y_1(t_0)\exp\left\{\int_{t_0}^{t} [\sigma_1(\tau) + c_1(\tau)g(\tau)]\,d\tau\right\} \tag{13}$$

Now, write $c_1 g = \langle c_1 g\rangle + \Delta(t)$, where $\langle c_1 g\rangle$ is the average over the entire time interval

$$\langle c_1 g\rangle = \frac{1}{t_f - t_0}\int_{t_0}^{t_f} c_1(\tau)g(\tau)\,d\tau \tag{14}$$

Similarly, write $\sigma_1(t) = \lambda_1 + \epsilon_1(t)$, where $\epsilon_1(t)$ represents the (not necessarily small) deviation of the instantaneous exponential rate σ_1 from its average value over the time interval. Then the solution for y_1, Eq. (13) becomes

$$y_1(t) = y_1(t_0)\exp[(\lambda_1 + \langle c_1 g\rangle)(t - t_0)] \times \exp\left(\int_{t_0}^{t} [\epsilon_1(\tau) + \Delta(\tau)]\,d\tau\right) \tag{15}$$

Now, the integral of $\epsilon_1(t) + \Delta(t)$ over the interval $t_0 \le t \le t_f$ is zero, so it is clear that the new regional Lyapunov exponent for this interval, for the "closed-loop state," is just

$$\lambda_1' = \lambda_1 + \langle c_1 g\rangle \tag{16}$$

Now, the closed-loop equations of motion for the other states become

$$\dot{y}_i = \sigma_i(t)y_i + c_i(t)g(t)y_1(t) \tag{17}$$

Regarding y_1 as a known function of time, this is a variable coefficient linear equation with a time-dependent forcing function. Its solution is found by elementary methods as

$$y_i(t) = y_i(t_0)\exp\left(\int_{t_0}^{t}\sigma_i(\tau)\,d\tau\right) + \int_{t_0}^{t}\exp\left(\int_{\tau}^{t}\sigma_i(\phi)\,d\phi\right)c_i(\tau)g(\tau)y_1(\tau)\,d\tau \tag{18}$$

By inspection, the homogeneous part is exponentially changing with the original exponential rate. The convolution integral above we will refer to as the zero-state solution $y_{i,ZS}(t)$, since it is the solution to Eq. (17) with zero initial condition.

We propose to determine the gain function by imposing the conditions that

$$\frac{1}{t_f - t_0}\int_{t_0}^{t_f} c_1(\tau)g(\tau)\,d\tau = \lambda_1' - \lambda_1 \tag{19}$$

(where λ_1' is now chosen a priori) and that the solution to Eq. (17) with $y_i(t_0) = 0$, the zero-state solutions $y_{i,ZS}$, should be zero at the final time. This reduces Eq. (18) at the final time to just

$$y_i(t_f) = y_i(t_0)\exp\left(\int_{t_0}^{t_f}\sigma_i(\tau)\,d\tau\right) = y_i(t_0)\exp\{\lambda_i(t_f - t_0)\} \tag{20}$$

In other words, to ensure that only λ_1 is changed, the zero-state solutions must vanish at the final time. The remaining homogeneous solution is then just the original uncontrolled response, at least at the end of the interval.

In Wiesel,[3] the unstable root was moved with a gain function that was a polynomial in time. In this paper we consider the problem of changing the unstable system exponent by a predetermined amount in an optimal fashion. Rewrite the modal system as

$$\dot{z}_1 = c_1(t)g(t)$$

$$\dot{z}_j = \sigma_j(t)z_j + c_j(t)g(t)\exp\left(\int_{t_0}^{t}\sigma_1(\tau)\,d\tau + z_1(t)\right) \tag{21}$$

$$j = 2, 3, \ldots, N$$

All of the above have zero initial conditions, ensuring that the solutions to the last $N - 1$ equations represent the zero-state solutions to the uncontrolled modes. To ensure that only λ_1 is changed, we must have final conditions

$$z_1(t_f) = \Delta\lambda_1(t_f - t_0)$$

$$z_j(t_f) = 0, \qquad j = 2, 3, \ldots, N \tag{22}$$

The first of these specifies the desired change in the unstable Lyapunov exponent, whereas the remainder are the modal decoupling conditions at the final time.

Now, determine the gain function by minimizing a quadratic cost function of the gain itself. That is, we will minimize

$$J = \int_{t_0}^{t_f} g^2(t)\,dt \tag{23}$$

in order to minimize the use of control to shift the unstable Lyapunov exponent. This optimal control problem has Eqs. (21) as constraints and Eqs. (22) as final conditions. Appending the constraint equations to the cost function leads to the control Hamiltonian

$$\mathcal{H}_c = g^2(t) + \gamma_1(t)c_1(t)g(t) + \sum_{j=2}^{N}\gamma_j(t)\left[\sigma_j(t)z_j(t) + c_j(t)g(t) \times \exp\left(\int_{t_0}^{t}\sigma_1(\tau)\,d\tau + z_1(t)\right)\right] \tag{24}$$

Here the $\gamma_k(t)$ are Lagrange multipliers. They have equations of motion $\dot{\gamma}_i = -\,\partial\mathcal{H}_c/\partial z_i$, which gives the first as

$$\dot{\gamma}_1 = -\sum_{j=2}^{N}\gamma_j(t)c_j(t)g(t)\exp\left(\int_{t_0}^{t}\sigma_1(\tau)\,d\tau + z_1(t)\right) \tag{25}$$

The equations of motion for the other Lagrange multipliers are trivial and immediately lead to the solutions $\gamma_j(t) = \gamma_j(t_0)\exp[-\int_{t_0}^{t}\sigma_j(\tau)\,d\tau]$. Since we have a full set of boundary conditions on the z_k at both the initial and final times, no boundary conditions for the Lagrange multipliers are known.

The optimality condition $\partial\mathcal{H}_c/\partial g = 0$ leads to a linear equation in $g(t)$, which is solved to yield the control law

$$g(t) = \frac{1}{2}\Bigg[\gamma_1(t)c_1(t) + \sum_{j=2}^{N}\gamma_j(t)c_j(t)\exp\bigg(\int_{t_0}^{t}\sigma_1(\tau)\,\mathrm{d}\tau + z_1(t)\bigg)\Bigg] \tag{26}$$

With all these portions of the problem available, we have posed the optimal modal control boundary value problem. A second-order gradient method was programmed to solve this problem numerically. Since full initial and final conditions are specified for the z_i, the problem becomes finding the initial values of the Lagrange multipliers $\gamma_i(t_0)$ to achieve the final boundary conditions in the $z_i(t_f)$. (Although the boundary value problem is linear in the γ_i, their influence on the final boundary conditions is nonlinear.) One solution to this problem is already known, since zero values for the Lagrange multipliers reproduce the uncontrolled case. So, beginning with this case, solutions can be found and extrapolated to ever larger values of $\Delta\lambda_1$ until acceptable performance is achieved.

With the controller in place, the closed-loop system is still a time-dependent linear problem. The closed-loop initial and final modal matrices $\mathcal{E}$ are unchanged from their values in the open-loop problem. This is easy to see from Eq. (12) for y_1, since it is decoupled from the rest of the problem. But our decision to decouple the other modes at t_f also leaves the initial and final modal matrices $\mathcal{E}(t_0)$ and $\mathcal{E}(t_f)$ unaltered. But within the interval $t_0 \le t \le t_f$ the $\mathcal{E}(t)$ matrix changes. Additional unstable modes can be controlled by iterating the above process, picking a second Lyapunov exponent to change, and designing a second controller on top of the closed-loop system for the first mode. Assuming satisfaction of a controllability condition, all of the modes could be controlled in this fashion. The decoupling conditions ensure that individual controllers will not interfere with each other.

IV. Re-Entry Controller

In this paper we consider, as an example, the problem of control of a shallow-angle re-entry from low Earth orbit. Figure 1 shows the variables used for this problem. We pick as coordinates the downrange distance X, measured along Earth's surface, and the altitude H. The kinetic energy of the spacecraft per unit mass is then $T = [\dot{H}^2 + \dot{X}^2(R+H)^2/R^2]/2$, whereas the two-body gravitational potential per unit mass is $V = -\mu/(R+H)$. Here R is the radius of Earth. The only other force acting on the spacecraft is air drag, which produces an acceleration (force per unit mass) of $\boldsymbol{a}_d = -B^*\rho v\boldsymbol{v}/2$, where $B^* = C_d A/m$ is the air drag surface loading factor, or ballistic coefficient. This dissipative force is handled by calculating generalized forces.

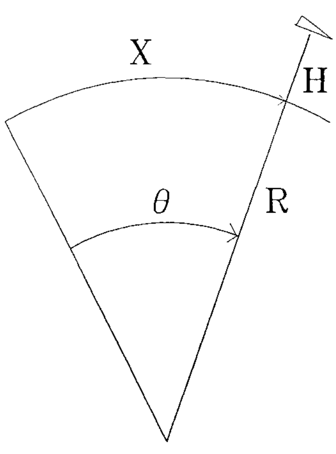

Fig. 1 Coordinate geometry for re-entry problem.

Using a Hamiltonian formulation, mainly because it directly gives first-order equations of motion, we have the momenta as

$$p_H = \dot{H}, \qquad p_X = \frac{(R+H)^2}{R^2}\dot{X} \tag{27}$$

whereas the Hamiltonian function is

$$\mathcal{H} = \frac{1}{2}\left(p_H^2 + \frac{R^2}{(R+H)^2}p_X^2\right) - \frac{\mu}{R+H} \tag{28}$$

The equations of motion are then given by

$$\dot{H} = \frac{\partial\mathcal{H}}{\partial p_H}, \qquad \dot{p}_H = -\frac{\partial\mathcal{H}}{\partial H} + Q_H$$
$$\dot{X} = \frac{\partial\mathcal{H}}{\partial p_X}, \qquad \dot{p}_X = -\frac{\partial\mathcal{H}}{\partial X} + Q_X \tag{29}$$

where the generalized forces are

$$Q_H = -\frac{1}{2}B^*\rho\left(p_H^2 + \frac{R^2}{(R+H)^2}p_X^2\right)^{\frac{1}{2}} p_H$$
$$Q_X = -\frac{1}{2}B^*\rho\left(p_H^2 + \frac{R^2}{(R+H)^2}p_X^2\right)^{\frac{1}{2}} p_X \tag{30}$$

We have used a simple exponential atmosphere $\rho = \rho_0\exp(H/H_0)$ for the air density ρ, where ρ_0 is the sea-level air density and H_0 is the atmospheric scale height.

The coordinates and momenta are assembled into a state vector,

$$X^T = \{X, H, p_X, p_H\} \tag{31}$$

We have chosen to control the trajectory by modulating the drag factor B^*. On the Delta Clipper vehicle, this would be done by extending large flat panels away from the side of the spacecraft. The control variable is $u = B^*$, and the control distribution matrix is then

$$B = \left\{0, 0, -\tfrac{1}{2}\rho v p_X, -\tfrac{1}{2}\rho v p_H\right\} \tag{32}$$

where

$$v = \left(p_H^2 + \frac{R^2}{(R+H)^2}p_X^2\right)^{\frac{1}{2}} \tag{33}$$

The author chose this problem as an example due to the highly time dependent nature of the system. Studying a similar problem, Roenneke and Cornwell[5] needed to design dozens of individual linear quadratic regulator (LQR) controllers for short-trajectory arcs, approximating each arc as a constant coefficient system. In this paper, the controller will be designed for the trajectory as a single unit.

Figure 2 shows the trajectory studied in this paper. Astrodynamic units were used for this problem, taking the equatorial radius of Earth R, the gravitational constant μ, and the mass of Earth as unity. Unit velocity is then 7.90536 km/s, whereas the time unit is 13.4468 min. The atmospheric parameters then take the values $H_0 = 1.1 \times 10^{-3}$ Earth radii, $\rho_0 = 5.3178 \times 10^{-5}$ Earth masses per Earth radius cubed. The air drag factor for the vehicle was $B^* = 1 \times 10^8$. Initial conditions for the trajectory are listed in Table 1. These correspond to a Hohmann transfer from a 250-km-altitude apogee toward a 50-km-altitude perigee, with re-entry beginning at 80 km altitude. Touchdown occurs at $t_f = 0.6063766$ time units, or about 8.1 min after atmospheric entry. The nominal drag factor is large enough that the vehicle approaches a terminal speed just before it reaches Earth's surface.

Table 2 gives the Lyapunov exponents for this trajectory. There is one unstable and three stable modes. Examination of the $\boldsymbol{e}_i$ vectors

Table 1 Trajectory initial conditions

X	0
H	1.25428×10^{-2}
p_X	1.01157186181
p_H	$-1.0930216899 \times 10^{-2}$

Table 2 Lyapunov exponents

Mode	λ_i
1	+7.5810168
2	−1.9279445
3	−13.0725430
4	−20.7439417

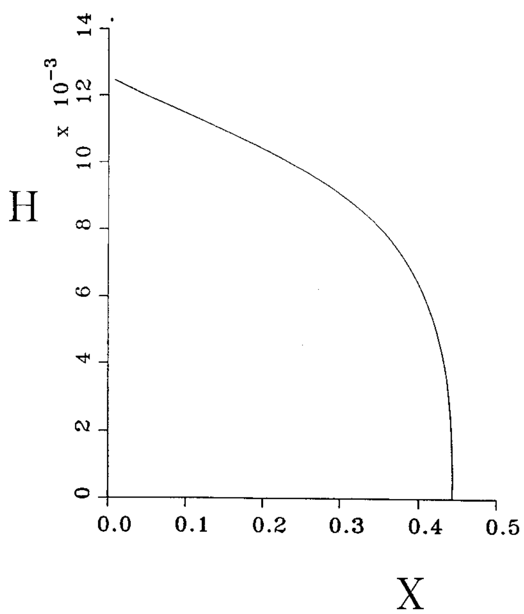

Fig. 2 Altitude H vs downrange distance X for trajectory studied in this paper.

shows that at atmospheric entry the unstable mode consists mainly of an error in entry altitude, whereas at touchdown it is virtually totally in the downrange direction. This is the instability that must be suppressed if the vehicle is to return to its launch pad. The second mode is marginally stable and at entry consists mostly of an error in downrange position X, with some component in the p_H direction. At landing, this mode becomes a mixture of p_X and p_H errors. The last two strongly stable modes start as errors mostly in the momentum components and at landing mode 3 consist of a mixture of momentum errors, whereas mode 4 becomes a nearly pure error in altitude at touchdown. None of the last three modes has any strong component in the downrange direction at landing. The modal matrix $\mathcal{E}(t)$ was checked throughout the trajectory to be sure that it remained nonsingular.

Figure 3 shows the functions $\sigma_i(t)$ for this trajectory. The regional Lyapunov exponents are the time averages of these functions,

$$\lambda_i = \frac{1}{t_f - t_0} \int_{t_0}^{t_f} \sigma_i(\tau)\, d\tau \tag{34}$$

but there are wide excursions about the mean values. In particular, the large negative spike in $\sigma_4(t)$ near touchdown is associated with convergence of all trajectories to a terminal velocity. This system is clearly and fundamentally a time-dependent system.

Figure 4 shows the closed-loop Lyapunov exponents as a function of the desired shift $\Delta\lambda_1$ in the unstable exponent. These values were calculated by integrating the Φ matrix for the closed-loop system (9) in physical units and then calculating the closed-loop Lyapunov exponents. Points in the figure are calculated exponents, whereas the straight lines are the predicted values. The success of the modal decoupling is shown by the lack of any shift in the uncontrolled modes. Figure 5 shows the optimal gain functions for several values of the root shift $\Delta\lambda_1$. Table 3 gives the initial Lagrange multipliers γ_i for the $\Delta\lambda_1 = -9$ case, which we will study further below.

Of course, this is a linear controller embedded in a nonlinear dynamic system. Thus the question of the region of validity of the system naturally arises. We have propagated trajectories adjacent to our reference trajectory with the controller for $\Delta\lambda_1 = -9$ operating

Table 3 Lagrange multipliers

γ_1	$+6.10415542 \times 10^{20}$
γ_2	$+4.48023194 \times 10^{20}$
γ_3	$-5.87782023 \times 10^{17}$
γ_4	$+3.04097467 \times 10^{16}$

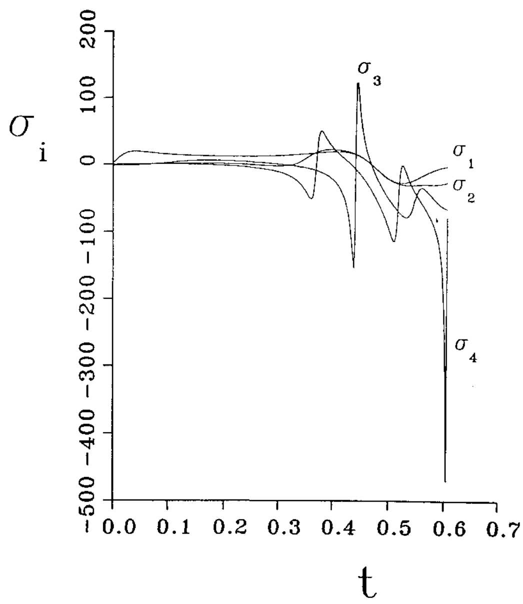

Fig. 3 Functions $\sigma_i(t)$ for re-entry trajectory.

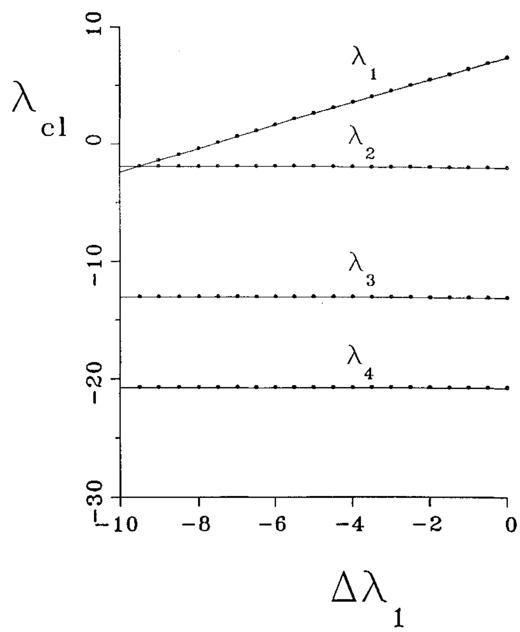

Fig. 4 Closed-loop Lyapunov exponents $\lambda_{\mathrm{cl},i}$ as function of shift $\Delta\lambda_1$ in unstable mode.

on adjacent trajectories. This gives a nominal closed-loop Lyapunov exponent of $\lambda_{1,\mathrm{cl}} = -1.419$. Initial conditions were chosen in the unstable modal variable $y_1(t_0)$ and converted to physical variables. Again, at the initial time the first mode is virtually an altitude error. The final modal amplitude $y_1(t_f)$ was found by numerical integration of the nonlinear system with the linear controller and converted to an "effective" Lyapunov exponent by

$$\lambda_{1,\mathrm{eff}} = \frac{1}{t_f - t_0} \ln \frac{y_1(t_f)}{y_1(t_0)} \tag{35}$$

This value is shown in Fig. 6 as a function of initial modal amplitude $y_1(t_0)$.

Because of the direction of our $\boldsymbol{e}_1(t_0)$, positive y_1 corresponds to a lower than expected initial re-entry altitude. The range of y_1 shown

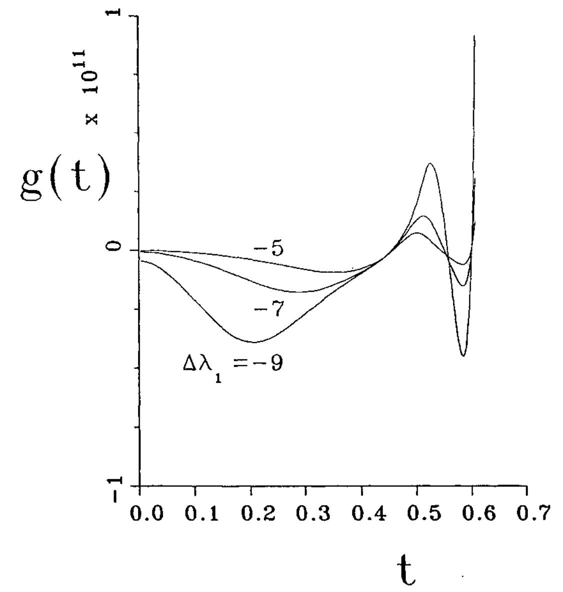

Fig. 5 Optimal gain functions $g(t)$ for several values of $\Delta\lambda_1$.

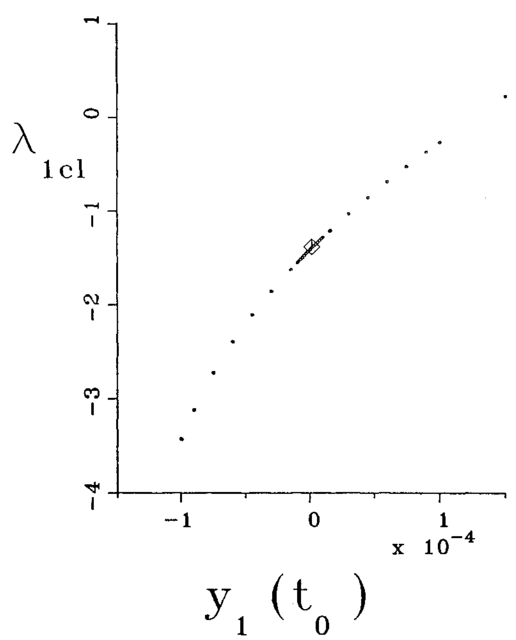

Fig. 6 Effective Lyapunov exponents for adjacent trajectories in nonlinear system with controller operating. Design point is shown at middle.

corresponds to a ±1-km error in the initial re-entry altitude. The desired closed-loop Lyapunov exponent is shown by the symbol at $y_1 = 0$. The controller is very stable for higher than expected trajectories, whereas it becomes marginally stable if the initial altitude is lower than expected. However, even this may be acceptable, since the uncontrolled system would amplify altitude errors by about a factor of 100, converting a −1-km altitude error at entry into a 100-km undershoot in the intended landing point. The controlled system produces virtually static errors in this condition. Notice from Figs. 4 and 6 that the product gy_1 is of order 4×10^6, or 4% of the original drag factor B^*. Even if this were optimistic by an order of magnitude, the required drag modulation should be well within the realm of possibility.

The fact that the controller is more stable for higher entry errors is desirable, but this is fortuitous. To be sure of landing on the desired point, either the drag control devices must be half deployed on the nominal trajectory, so that the drag can be decreased when the controller commands it, or else the trajectory must be biased to deliberately overshoot the desired landing point. This bias would take the form of higher than nominal initial entry altitude. This is just the condition in which our controller is very stable.

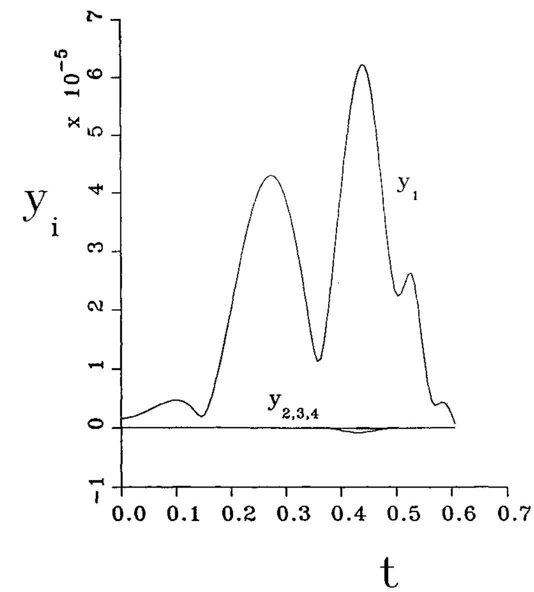

Fig. 7 Modal variables $y_i(t)$ for closed-loop system. Forced oscillations in uncontrolled modes are negligible.

Our controller ensures that the modes are decoupled only at the final time. Although the homogeneous part of Eq. (17) is also unaltered by the controller, the uncontrolled modes suffer a forced response due to the action of the controller. This forced response should be zero only at the final time. It is also interesting to check the trajectory deviations in the controlled mode y_1. Figure 7 shows the modal variables against time during one simulation. The initial y_1 displacement was 10 m, and this grows to about 400 m before decaying to the predicted value at t_f. Forced responses in the other modal variables are insignificant. The large excursions in y_1 occur while the vehicle is still quite high in the atmosphere, so they pose no danger.

V. Conclusions

We have shown that the modal decomposition of time-dependent variational equations leads to both stability information and insight into exactly how the system is unstable. Feedback control can be used to deterministically alter any or all of the Lyapunov exponents. Since placing even one pole requires a gain function, there is considerable room for optimization even when only a single root is being moved. In this paper we have minimized the mean-square gain over the trajectory, but other optimization criteria are, of course, possible.

As a pole placement method for time-dependent linear systems, one of the great advantages of our method is that there are no restrictions on the character of the reference trajectory. A realistic re-entry would occur in the real atmosphere, not an exponential atmosphere. Lift could be used to stay high in the atmosphere longer, and planned lateral excursions off the flight path would be included to remove the initial downrange overshoot of the trajectory. Even with all these factors included, the present method could still separate the dynamic modes and calculate feedback gains to control unstable modes.

Acknowledgments

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